Signals Through the Lens of Projection

Compression, Analytic Signals, and the Geometry of $L^2(\mathbb{T})$

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One Geometry, Many Signals

Functional analysis gives a geometric view of signals:

signal $\in$ Hilbert space.

Two very important things we do with signals:

- Compression: keep the components that matter (DCT, Fourier).
- AM-FM analysis: keep the *causal* or *positive-frequency* part.

Both are instances of the same operation: Choose a subspace. Project onto it.

Compression = finite-dimensional projection. Analytic signal = projection onto the Hardy space H^2*. Same mathematics. Different engineering goals.*

$L^2(\mathbb{T})$: The Space of Signals

$$L^2(\mathbb{T}) = \left\{ f : \mathbb{T} \to \mathbb{C} \;\middle|\; \|f\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(e^{i\theta})|^2 \, d\theta < \infty \right\}.$$

The variable $e^{i\theta}$ represents discrete-time frequency. We will later show (via Parseval) that this norm corresponds to signal energy.

Inner product and induced geometry

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})\,\overline{g(e^{i\theta})}\,d\theta.$$

Proposition. The mapping $\langle\cdot,\cdot\rangle$ defines a Hermitian inner product on $L^2(\mathbb{T})$, hence a norm: $\|f\|_2 = \sqrt{\langle f, f\rangle}$.

Proof (properties).

1. *Conjugate symmetry.* $\overline{\langle f, g\rangle} = \frac{1}{2\pi}\int \overline{f(e^{i\theta})}\,g(e^{i\theta})\,d\theta = \langle g, f\rangle$.
2. *Linearity in the first argument.* For $a, b \in \mathbb{C}$, $\langle af_1 + bf_2, g\rangle = a\langle f_1, g\rangle + b\langle f_2, g\rangle$, by linearity of the integral.
3. *Positive definiteness.* $\langle f, f\rangle = \frac{1}{2\pi}\int |f|^2 \geq 0$, and $\langle f, f\rangle = 0$ iff $f = 0$ a.e.

Orthogonality ($\langle f, g\rangle = 0$) means zero correlation across all frequencies.

Completeness of $L^2(\mathbb{T})$

Theorem. $(L^2(\mathbb{T}), \|\cdot\|_2)$ is complete.

$$(f_n) \text{ Cauchy in } L^2(\mathbb{T}) \quad \Longrightarrow \quad \exists f \in L^2(\mathbb{T}) : \ \|f_n - f\|_2 \to 0.$$

Proof (sketch).

1. Since (f_n) is Cauchy, for every $\varepsilon > 0$ there exists N such that
$$\|f_n - f_m\|_2 < \varepsilon \quad \text{for } n, m \geq N.$$
2. Fix $m \geq N$. Then the sequence $\|f_n - f_m\|_2$ tends to 0 as $n \to \infty$.
3. Therefore (f_n) is a Cauchy sequence centered at f_m and the limit
$$f := \lim_{n\to\infty} f_n$$
exists in the $\mathbf{L^2}$ norm.
4. Since $\|f_n - f\|_2 \to 0$ and each $f_n \in L^2(\mathbb{T})$, the limit f also lies in $L^2(\mathbb{T})$.

Cauchy sequences of signals always converge inside L^2. No signal "escapes" the space.

Separability of $L^2(\mathbb{T})$

Theorem. $L^2(\mathbb{T})$ is separable. The exponentials $\{e^{in\theta}\}_{n\in\mathbb{Z}}$ form a countable orthonormal basis.

$$f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \widehat{f}(n)\, e^{in\theta}, \qquad \widehat{f}(n) = \langle f, e^{in\theta}\rangle.$$

Proof (sketch).

1. Orthogonality and unit norm follow directly: $\langle e^{in\theta}, e^{im\theta}\rangle = \delta_{nm}$.
2. Trigonometric polynomials $p(\theta) = \sum_{n=-N}^{N} c_n e^{in\theta}$ are dense in $L^2(\mathbb{T})$.
3. Therefore every $f \in L^2(\mathbb{T})$ can be approximated in L^2 by such polynomials, giving the expansion above.

The space has a countable orthonormal basis. This makes Fourier analysis computable.

Parseval's Theorem

Theorem. For any $f \in L^2(\mathbb{T})$,

$$\|f\|_2^2 = \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^2, \qquad \widehat{f}(n) = \langle f, e^{in\theta}\rangle.$$

Proof (sketch).

1. Expand f in the orthonormal basis:

$$f = \sum_n \widehat{f}(n) e^{in\theta}.$$

2. By orthonormality,

$$\|f\|_2^2 = \langle f, f\rangle = \sum_n |\widehat{f}(n)|^2.$$

3. This is exactly the Pythagorean theorem in an infinite-dimensional Hilbert space.

Energy in time = energy in frequency. Fourier transform is unitary.

The Projection Theorem

Theorem (Hilbert Projection): Let H be a Hilbert space and $M \subset H$ a closed subspace. For any $x \in H$:

1. **Existence & Uniqueness:** There exists a unique element $m \in M$ such that

$$\|x - m\| = \inf_{y \in M} \|x - y\|.$$

2. **Orthogonality Characterization:** The minimizer m is uniquely determined by

$$x - m \perp M \quad \iff \quad \langle x - m, y \rangle = 0 \text{ for all } y \in M.$$

$P_M : H \to M$ *is linear, bounded, self-adjoint, and idempotent. Foundation of optimal approximation.*

Image Compression = Projection

$I \in \mathbb{R}^{N\times N}$, $\{\phi_{u,v}\}$ orthonormal DCT basis.

$$\hat{I}(u,v) = \langle I, \phi_{u,v}\rangle, \quad \|I\|_F^2 = \sum_{u,v} |\hat{I}(u,v)|^2$$

$$P_M(I) = \sum_{(u,v)\in M} \hat{I}(u,v)\phi_{u,v}$$

JPEG = project onto $\operatorname{span}\{\phi_{u,v} : (u,v) \in M\}$*.*
Optimal by Projection Theorem.

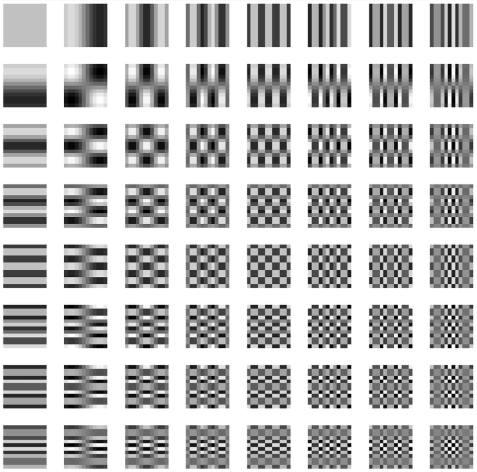


Figure 1: DCT basis images forming the DCT dictionary. [1]

Compression Error Theorem

Theorem. For any image I and any retained coefficient set M,

$$\|I - P_M(I)\|_F^2 = \sum_{(u,v)\notin M} |\widehat{I}(u,v)|^2.$$

Why this holds.

- DCT basis vectors $\{\phi_{u,v}\}$ are orthonormal.
- Projection $P_M(I)$ keeps coefficients in M and zeros the rest.
- The error lives entirely in the discarded directions.

Error = energy outside the chosen subspace. Compression works because natural images have rapidly decaying DCT energy.

Check the notebook for a visualization of this projection and DCT Energy!

Quantization = Nearest-Point Projection

$$Q_\Delta(x) \;=\; \Delta\,\mathrm{round}(x/\Delta).$$

Geometric interpretation:

- The set $\Lambda_\Delta = \Delta\mathbb{Z}^n$ is a *lattice*: a grid of representable points.
- Quantization chooses the lattice point nearest to x in Euclidean distance.
- So Q_Δ is a *metric projection* onto the discrete set Λ_Δ.

Fine $\Delta \to$ dense lattice $\to$ small error.

Coarse $\Delta \to$ sparse lattice $\to$ large error.

Distortion is simply the distance to the nearest lattice point. Rate controls lattice density.

The Time Domain: Causality = Analyticity

- **Causality:** A discrete-time system is *causal* if $h[n] = 0$ for $n < 0$.

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n}$$

  has only nonpositive powers of z.

- **Analyticity:** A function $F(z)$ is *analytic* in a region Ω if it can be written as a convergent power series

$$F(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

  i.e. complex differentiable at every $z \in \Omega$.

- **Equivalence:** Causality $\iff$ $H(z)$ analytic inside the unit disk $\mathbb{D}$. Filtering $\iff$ projection of $L^2(\mathbb{T})$ onto the analytic subspace H^2.

Orthogonal Decomposition Theorem

Theorem. $L^2(\mathbb{T}) = H^2 \oplus \overline{H_0^2}$, where

$$H^2 = \{f : \hat{f}_n = 0 \text{ for } n < 0\}, \qquad \overline{H_0^2} = \{f : \hat{f}_n = 0 \text{ for } n \geq 0\}.$$

Why this decomposition holds.

- The exponentials $e^{in\theta}$ form an orthonormal basis of $L^2(\mathbb{T})$.
- The index sets $\{n \geq 0\}$ and $\{n < 0\}$ produce two orthogonal subspaces.
- **Closedness of H^2:** Let $f_k \in H^2$ and $f_k \to f$ in $L^2(\mathbb{T})$. For any $n < 0$,

  $$\hat{f}_{k,n} = \langle f_k, e^{in\theta} \rangle = 0.$$

  Fourier coefficients depend continuously on the L^2 norm, so

  $$\hat{f}_{k,n} \longrightarrow \hat{f}_n.$$

  Since the left-hand side is identically 0, we get $\hat{f}_n = 0$ for all $n < 0$. Hence $f \in H^2$, so H^2 is closed.

Every signal in $L^2(\mathbb{T})$ splits uniquely into positive- and negative-frequency parts.

Riesz Projection Theorem

Definition. For $f(\theta) = \sum_{n\in\mathbb{Z}} \hat{f}_n e^{in\theta}$,

$$P_{H^2} f = \sum_{n=0}^{\infty} \hat{f}_n e^{in\theta}.$$

Properties.

$$P_{H^2}^2 = P_{H^2}, \qquad P_{H^2}^* = P_{H^2}, \qquad \|P_{H^2}\| = 1.$$

Why it works. The functions $\{e^{in\theta} : n \geq 0\}$ form an ONB for H^2, so truncating negative coefficients is exactly the orthogonal projection.

Causal part = positive frequencies. Riesz projection is stable and energy-nonincreasing.

Definition (Fourier multiplier). For a periodic signal

$$f(\theta) = \sum_{n \in \mathbb{Z}} \hat{f}_n e^{in\theta},$$

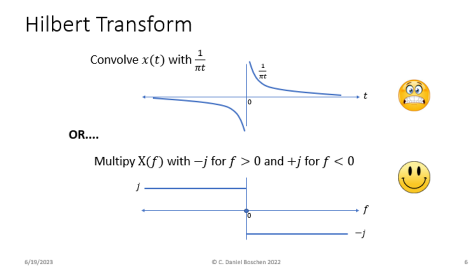
the Hilbert transform is

$$Hf(\theta) = \sum_{n \in \mathbb{Z}} \big(-i\,\mathrm{sgn}(n)\big)\,\hat{f}_n e^{in\theta}.$$

Interpretation.

- positive frequencies are rotated by -90°,
- negative frequencies are rotated by $+90^\circ$,
- the DC term is untouched.

Connection to convolution. On $\mathbb{R}$, this operator equals convolution with $1/(\pi t)$ in the principal-value sense.



The Analytic Signal: Strip Away Negative Frequencies

Definition. The analytic signal of f is $f_a(\theta) = f(\theta) + i\,Hf(\theta)$.

Fourier-domain expression. If $f(\theta) = \sum \hat{f}_n e^{in\theta}$, then $f_a(\theta) = \sum_{n\geq 0} 2\,\hat{f}_n e^{in\theta}$.

Key Points.

- f_a contains only nonnegative frequencies.
- f_a is the boundary value of a function analytic in the disk (i.e. in H^2).
- $f = \mathrm{Re}(f_a)$.

Why this is a boundary value. Since $f_a(\theta) = \sum_{n\geq 0} 2\hat{f}_n e^{in\theta}$ has only nonnegative Fourier coefficients, the power series $F(z) = \sum_{n\geq 0} 2\hat{f}_n z^n$ is analytic on $|z| < 1$ and satisfies $F(re^{i\theta}) \to f_a(\theta)$ as $r \to 1^-$.

$f_a = 2\,P_{H^2}(f)$. *Analytic signal = the H^2 projection written in the time domain.*

Why "Analytic" Signal?

Fact. For any real signal $x(t)$, the analytic signal

$$x_a(t) = x(t) + iH[x](t)$$

has Fourier transform

$$X_a(\omega) = \begin{cases} 2X(\omega), & \omega > 0, \\ 0, & \omega < 0. \end{cases}$$

Theorem. A function F is boundary data of a Hardy-space analytic function in the unit disk (or upper half-plane) *iff* its spectrum vanishes on negative frequencies.

$$X_a(\omega) = 0 \text{ for } \omega < 0 \quad \Longleftrightarrow \quad x_a \in H^2(\text{upper half-plane}).$$

Interpretation. "Analytic" refers to analyticity in the *complex-frequency* domain, not the time domain.

Envelope and Instantaneous Frequency

Let the analytic signal be

$$f_a(\theta) = A(\theta)\, e^{i\phi(\theta)},$$

where

$$A(\theta) = |f_a(\theta)|, \qquad \phi(\theta) = \arg f_a(\theta).$$

Envelope. $A(\theta) = \sqrt{f(\theta)^2 + (Hf(\theta))^2}$.

Instantaneous frequency. $\omega(\theta) = \phi'(\theta)$.

Interpretation.

- $A(\theta)$ tracks the amplitude modulation of the signal.
- $\omega(\theta)$ captures local frequency variation.
- Both are only well-defined once negative frequencies are removed.

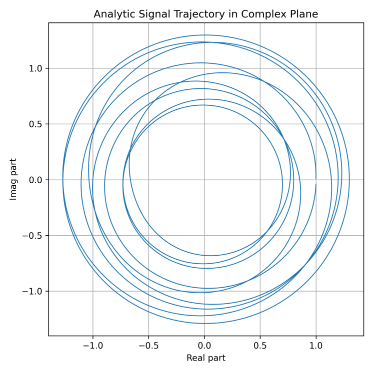
Hilbert transform = machinery that makes AM-FM decomposition unique.

Visualizing the Analytic Signal

Consider a real AM-FM signal $x(t) = A(t)\cos(\phi(t))$. Its analytic version $x_a(t) = x(t) + i\,H[x](t)$ has the polar form $x_a(t) = A(t)\,e^{i\phi(t)}$.

What the complex plot shows:

- $\Re(x_a(t))$ = original real signal.
- $\Im(x_a(t))$ = its Hilbert transform.
- For slowly-varying AM–FM signals $x(t) = A(t)\cos(\phi(t))$, the magnitude $|x_a(t)|$ approximates $A(t)$ and the argument $\arg(x_a(t))$ approximates $\phi(t)$. $x(t)$ and $H[x](t)$ evolve jointly.



$$x_a(t) = \big(1 + 0.3\sin(4\pi t)\big)\ \exp\Big(i\big[10\pi t + 4\pi\sin(2\pi t)\big]\Big).$$

Analytic signal = unique complex signal with no negative frequencies.

Two Definitions. One Object.

Engineering view.

$$x_a(t) = x(t) + i\,H[x](t)$$

A quadrature signal with no negative frequencies.

Functional analytic view.

$$x_a \in H^2$$

The boundary trace of an analytic function in the upper half-plane (or the unit disk).

Conclusion. These two constructions coincide:

$$x_a = 2\,P_{H^2}(x).$$

Hilbert transform is not an add on. It is the real line shadow of the Riesz projection.

Projections Everywhere

Compression.

$$P_M(I) = \text{best approximation in the DCT subspace.}$$

Analytic Signal.

$$x_a = 2\,P_{H^2}(x).$$

Interpretation. The same Hilbert-space geometry underlies:

- keeping low-frequency image content,
- keeping positive-frequency signal content.

Different engineering goals, same mathematical skeleton.

Closing

Whether compressing images or extracting AM-FM structure, we are always projecting onto the right subspace.

Hilbert spaces justify why these operations are stable, optimal, and the only reasonable tools we have.

Functional analysis is the quiet backbone of signal processing.

Camelia Florea, Mihaela Gordan, Bogdan Orza, and Aurel Vlaicu.
Compressed domain computationally efficient processing scheme for jpeg image filtering.
Advanced Engineering Forum, 8-9:480–489, 06 2013.

Najah F. Ghalyan, Asok Ray, and William Kenneth Jenkins.
A concise tutorial on functional analysis for applications to signal processing.
Sci, 4(4):40, 2022.

erwin kreyszig.
introductory functional analysis with applications.
wiley, 1978.